

# New Properties of Fourier Series and Riemann Zeta Function

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## Abstract

We establish the mapping relations between analytic functions and periodic functions using the abstract operators  $\cos(h\partial_x)$  and  $\sin(h\partial_x)$ , including the mapping relations between power series and trigonometric series, and by using such mapping relations we obtain a general method to find the sum function of a trigonometric series. According to this method, if each coefficient of a power series is respectively equal to that of a trigonometric series, then if we know the sum function of the power series, we can obtain that of the trigonometric series, and the non-analytical points of which are also determined at the same time, thus we obtain a general method to find the sum of the Dirichlet series of integer variables, and derive several new properties of  $\zeta(2n+1)$ .

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## 1 Introduction

The trigonometric series, especially the Fourier series, is of great importance in both mathematics and physics, and expanding periodic functions into Fourier series has become a very mature theory. During the early years of last century, people realized the significance of the inverse problem, which is how to find the sum function of a certain Fourier series. So the question is: can we use the sum function of power series to obtain that of trigonometric series? As we know, the domain of functions expressed by trigonometric series can be extended into the entire number axis with the existence of denumerable non-analytical points, thus functions expressed by trigonometric series are piecewise analytic periodic functions. If a trigonometric series converges to an analytic sum function in a certain interval, then it is quite natural that there is a mapping relation between a periodic function and an analytic function, though the two endpoints of the interval generally are its non-analytical points. Therefore, we can use the sum of power series to obtain that of corresponding trigonometric series, converting the research focus from trigonometric series to power series. For a long time, the first author has realized the fact that the abstract operators  $\cos(h\partial_x)$  and  $\sin(h\partial_x)$  can express the mapping relations between periodic functions and analytic functions more distinctly, which has become a significant tool to obtain the sum function of trigonometric series. Why the abstract operators  $\cos(h\partial_x)$  and  $\sin(h\partial_x)$  can establish such relations? That is because this kind of operators is also a kind of functions, containing the duality between periodic functions and linear operators. By using such a duality, the authors have preliminarily established theories of partial differential equations of abstract operators in reference [1]-[5].

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What is an abstract operator? The operator  $f(t, \partial_x)$  is generally interpreted as a Taylor expansion, called the infinite order operator. However, the need to consider its disk of convergence is a major constraint of its broad applications. Therefore, the first author has defined the operator  $f(t, \partial_x)$  as  $f(t, \partial_x)e^{ax} = f(t, a)e^{ax}$  in 1997. Each operator  $f(t, \partial_x)$  has a set of algorithms without the need to use its Taylor expansion, and the first author has also provided a method to determine such kind of algorithms. In this sense, the operator  $f(t, \partial_x)$  is known as the abstract operator in reference [1]. However, the concept of abstract operators has not yet been spread adequately. As a result, it is easy to mistake the abstract operator for the infinite order operator by the similar symbol they are using. In fact, only several simple abstract operators can be expanded into Taylor series under certain conditions, such as  $\cos(h\partial_x)$  and  $\sin(h\partial_x)$ . However, the particular method we use to calculate is not by using their Taylor expansions but the certain algorithms of  $\cos(h\partial_x)$  and  $\sin(h\partial_x)$  established by the authors in their former published papers, such as the formulas (4), (7), (20), (13) and (14). Although there may be several non-analytical points in the process of calculation, making the results at these points not tenable, yet we will not encounter the infinite series, which benefits us a lot by avoiding the need to consider the disk of convergence.

The abstract operators  $\cos(h\partial_x)$  and  $\sin(h\partial_x)$  have exclusive advantages in symbol expressions, and the summation method of Fourier series established from which can be directly extended into multiple Fourier series.

By using this summation method of Fourier series, the abstract operators can also have significant applications in the Riemann Zeta function  $\zeta(m)$  of the analytic number theory. The Riemann Zeta function  $\zeta(s)$  defined usually by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\Re(s) > 1). \quad (1)$$

In 1735, Euler proved that for an arbitrary even number  $2K > 0$ ,  $\zeta(2K) = a_{2K}\pi^{2K}$ , where  $a_{2K}$  is a rational number. However, for all odd numbers  $2K + 1$ , the arithmetic properties of  $\zeta(2K + 1)$  are still unknown. As the summation method of trigonometric series of abstract operators is quite suitable to find the sum of the Dirichlet series of integer variables including the Zeta function, we take an important step in studying the arithmetic properties of  $\zeta(2K + 1)$ .

## 2 Basic formulas of abstract operators

When acting on elementary functions, the abstract operators  $\cos(h\partial_x)$  and  $\sin(h\partial_x)$  have complete basic formulas as differential operations, now we are going to use the algorithms of abstract operators in reference [1] or [4] to establish these formulas.

Firstly, according to the definition of abstract operators, we have

$$\cos(h\partial_x)e^{bx} = \cos(bh)e^{bx}, \quad \sin(h\partial_x)e^{bx} = \sin(bh)e^{bx}. \quad (2)$$

Where  $bx = b_1x_1 + b_2x_2 + \cdots + b_nx_n$ ,  $bh = b_1h_1 + b_2h_2 + \cdots + b_nh_n$ .

According to Theorem 2 in reference [1], namely

$$\exp(ih\partial_x)f(x) = f(x + ih),$$

and Theorem 3:

$$\cos(h\partial_x)f(x) = \Re[f(x + ih)], \quad \sin(h\partial_x)f(x) = \Im[f(x + ih)],$$

we have

$$\begin{aligned}
\cos(h\partial_x) \cos bx &= \cosh(bh) \cos bx, \\
\cos(h\partial_x) \sin bx &= \cosh(bh) \sin bx, \\
\sin(h\partial_x) \cos bx &= -\sinh(bh) \sin bx, \\
\sin(h\partial_x) \sin bx &= \sinh(bh) \cos bx.
\end{aligned} \tag{3}$$

Based on (3), and by using Theorem 6 in reference [1]:

$$\begin{aligned}
\sin(h\partial_x) \frac{u}{v} &= \frac{\cos(h\partial_x)v \cdot \sin(h\partial_x)u - \sin(h\partial_x)v \cdot \cos(h\partial_x)u}{(\cos(h\partial_x)v)^2 + (\sin(h\partial_x)v)^2}, \\
\cos(h\partial_x) \frac{u}{v} &= \frac{\cos(h\partial_x)v \cdot \cos(h\partial_x)u + \sin(h\partial_x)v \cdot \sin(h\partial_x)u}{(\cos(h\partial_x)v)^2 + (\sin(h\partial_x)v)^2},
\end{aligned} \tag{4}$$

we have

$$\begin{aligned}
\cos(h\partial_x) \tan bx &= \frac{\sin(2bx)}{\cosh(2bh) + \cos(2bx)}, \\
\cos(h\partial_x) \cot bx &= \frac{\sin(2bx)}{\cosh(2bh) - \cos(2bx)}, \\
\sin(h\partial_x) \tan bx &= \frac{\sinh(2bh)}{\cosh(2bh) + \cos(2bx)}, \\
\sin(h\partial_x) \cot bx &= \frac{\sinh(2bh)}{\cos(2bx) - \cosh(2bh)}.
\end{aligned} \tag{5}$$

For secant and cosecant functions, similarly to (5), we have

$$\begin{aligned}
\cos(h\partial_x) \sec bx &= \frac{2 \cosh(bh) \cos bx}{\cosh(2bh) + \cos(2bx)}, \\
\cos(h\partial_x) \csc bx &= \frac{2 \cosh(bh) \sin bx}{\cosh(2bh) - \cos(2bx)}, \\
\sin(h\partial_x) \sec bx &= \frac{2 \sinh(bh) \sin bx}{\cosh(2bh) + \cos(2bx)}, \\
\sin(h\partial_x) \csc bx &= \frac{2 \sinh(bh) \cos bx}{\cos(2bx) - \cosh(2bh)}.
\end{aligned} \tag{6}$$

According to the following theorem:

**Theorem 1.** [4] If  $y = f(bx) \in J$  (set of analytic functions) is the inverse function of  $bx = g(y)$ , namely  $g(f(bx)) = bx$ , then  $\sin(h\partial_x)f(bx)$  (denoted by  $Y$ ) and  $\cos(h\partial_x)f(bx)$  (denoted by  $X$ ) can be determined by the following set of equations:

$$\begin{aligned}
\cos\left(Y \frac{\partial}{\partial X}\right) g(X) &= bx \\
\sin\left(Y \frac{\partial}{\partial X}\right) g(X) &= bh
\end{aligned} \tag{7}$$

$x \in \mathbb{R}^n, \quad h \in \mathbb{R}_n,$

we can derive basic formulas of the corresponding inverse function:

$$\begin{aligned}
\cos(h\partial_x) \ln(bx) &= \ln((bx)^2 + (bh)^2)^{1/2}, \\
\sin(h\partial_x) \ln(bx) &= \operatorname{arccot} \frac{bx}{bh}.
\end{aligned} \tag{8}$$

$$\begin{aligned}
\sin(h\partial_x) \arctan bx &= \frac{1}{2} \tanh^{-1} \frac{2bh}{1 + (bx)^2 + (bh)^2}, \\
\cos(h\partial_x) \arctan bx &= \frac{1}{2} \arctan \frac{2bx}{1 - (bx)^2 - (bh)^2}.
\end{aligned} \tag{9}$$

$$\begin{aligned}
\sin(h\partial_x) \operatorname{arccot} bx &= -\frac{1}{2} \coth^{-1} \frac{1 + (bx)^2 + (bh)^2}{2bh}, \\
\cos(h\partial_x) \operatorname{arccot} bx &= \frac{1}{2} \operatorname{arccot} \frac{(bx)^2 + (bh)^2 - 1}{2bx}.
\end{aligned} \tag{10}$$

**Proof.** Here we only give the detailed proof of (8) and (9). According to (2) and (7), we have

$$\begin{aligned}
e^X \cos Y &= bx & X &= \cos(h\partial_x) \ln(bx), \\
e^X \sin Y &= bh & Y &= \sin(h\partial_x) \ln(bx).
\end{aligned}$$

By solving this set of equations we have (8), and according to (5) and (7), we have

$$\begin{aligned}
\frac{\sin 2X}{\cosh 2Y + \cos 2X} &= bx & X &= \cos(h\partial_x) \arctan bx, \\
\frac{\sinh 2Y}{\cosh 2Y + \cos 2X} &= bh & Y &= \sin(h\partial_x) \arctan bx.
\end{aligned}$$

By solving this set of equations we have

$$1 + (bx)^2 + (bh)^2 = 1 + \frac{\sin^2 2X + \sinh^2 2Y}{(\cosh 2Y + \cos 2X)^2} = \frac{2 \cosh 2Y}{\cosh 2Y + \cos 2X}.$$

According to the second expression of this set of equations, we have  $\cosh 2Y + \cos 2X = (\sinh 2Y)/bh$ , and by substituting it into the above expression, we have

$$1 + (bx)^2 + (bh)^2 = \frac{2bh \cosh 2Y}{\sinh 2Y} \quad \text{or} \quad \tanh 2Y = \frac{2bh}{1 + (bx)^2 + (bh)^2}.$$

Thus the first expression of (9) is proved, similarly we can prove the second one.

For hyperbolic and inverse hyperbolic functions, we can also derive the corresponding basic formulas. For instance, correspondingly to (3), we have

$$\begin{aligned}
\cos(h\partial_x) \cosh bx &= \cos(bh) \cosh bx, \\
\cos(h\partial_x) \sinh bx &= \cos(bh) \sinh bx, \\
\sin(h\partial_x) \cosh bx &= \sin(bh) \sinh bx, \\
\sin(h\partial_x) \sinh bx &= \sin(bh) \cosh bx.
\end{aligned} \tag{11}$$

Irrational functions can be considered as the inverse functions of rational functions. For instance,  $(bx)^{1/2}$  is the inverse function of  $(bx)^2$ , thus similarly we have

$$\begin{aligned}\sin(h\partial_x)(bx)^{1/2} &= \sqrt{\frac{\sqrt{(bx)^2 + (bh)^2} - bx}{2}}, \\ \cos(h\partial_x)(bx)^{1/2} &= \sqrt{\frac{\sqrt{(bx)^2 + (bh)^2} + bx}{2}}.\end{aligned}\tag{12}$$

To calculate concisely, the algorithms of products and composite functions in reference [1] are listed as follows, while the algorithm of quotients has already been given in (4):

$$\begin{aligned}\sin(h\partial_x)(vu) &= \cos(h\partial_x)v \cdot \sin(h\partial_x)u + \sin(h\partial_x)v \cdot \cos(h\partial_x)u, \\ \cos(h\partial_x)(vu) &= \cos(h\partial_x)v \cdot \cos(h\partial_x)u - \sin(h\partial_x)v \cdot \sin(h\partial_x)u.\end{aligned}\tag{13}$$

$$\begin{aligned}\cos(h\partial_x)f(g(x)) &= \cos\left(Y\frac{\partial}{\partial X}\right)f(X), \\ \sin(h\partial_x)f(g(x)) &= \sin\left(Y\frac{\partial}{\partial X}\right)f(X).\end{aligned}\tag{14}$$

Where  $x \in \mathbb{R}^n$ ,  $h \in \mathbb{R}_n$ ,  $X = \cos(h\partial_x)g(x)$ ,  $Y = \sin(h\partial_x)g(x)$ .

### 3 Summation method of trigonometric series

**Theorem 2.** Let  $S(t) \in J$  be the sum function of the power series  $\sum_{n=0}^{\infty} a_n t^n$ ,  $f(x) \in L^2[a, b]$  be the sum function of the corresponding cosine series, and  $g(x) \in L^2[a, b]$  be that of the corresponding sine series, namely

$$\begin{aligned}S(t) &= \sum_{n=0}^{\infty} a_n t^n, \quad t \in \mathbb{R}^1, \quad 0 \leq t \leq r, \quad 0 < r < +\infty. \\ f(x) &= \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{c}, \\ g(x) &= \sum_{n=0}^{\infty} a_n \sin \frac{n\pi x}{c}, \quad x \in \mathbb{R}^1, \quad a < x < b,\end{aligned}$$

then we have the following mapping relations:

$$\begin{aligned}f(x) &= \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S(e^z) \Big|_{z=0}, \\ g(x) &= \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S(e^z) \Big|_{z=0}, \quad z \in \mathbb{R}^1.\end{aligned}\tag{15}$$

And the endpoints  $a$  and  $b$  of the interval  $a < x < b$  are non-analytical points (singularities) of Fourier series, which can be uniquely determined by the detailed computation of the right-hand side of (15).

**Proof.** By substituting  $S(e^z) = \sum_{n=0}^{\infty} a_n e^{nz}$  into (15), we can prove Theorem 2.

The sum function of infinite power series can be an elementary function, which in most cases can be expressed as the definite integral of an elementary function, thus in the application of Theorem 2, the following theorem can be particularly useful:

**Theorem 3.** Let  $S(x) \in J$  be an arbitrary analytic function integrable in the interval  $[0, 1]$ , then we have

$$\begin{aligned} & \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \int_0^{e^z} S(e^z) de^z \Big|_{z=0} \\ &= \int_0^1 S(\xi) d\xi - \frac{\pi}{c} \int_0^x \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) [S(e^z) e^z] \Big|_{z=0} dx. \end{aligned} \quad (16)$$

$$\sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \int_0^{e^z} S(e^z) de^z \Big|_{z=0} = \frac{\pi}{c} \int_0^x \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) [S(e^z) e^z] \Big|_{z=0} dx. \quad (17)$$

**Proof.** (16) and (17) are operator formulas. According to the analytic continuous fundamental theorem in reference [1], we only need to prove this set of formulas when  $S(x) = x^n$ ,  $n \in \mathbb{N}_0$ , this is obvious.

**Theorem 4.** Let  $S(x) \in J$  be an arbitrary analytic function integrable in the interval  $[0, 1]$ , if  $\int_0^t S(t) dt$  is the sum function of the power series  $\sum_{n=1}^{\infty} a_n t^n$ , let  $f(x) \in L^2[a, b]$  be the sum function of the corresponding cosine series, and  $g(x) \in L^2[a, b]$  be that of the corresponding sine series, namely

$$\begin{aligned} \int_0^t S(t) dt &= \sum_{n=1}^{\infty} a_n t^n, \quad t \in \mathbb{R}^1, \quad 0 \leq t \leq r, \quad 0 < r < +\infty, \\ f(x) &= \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}, \\ g(x) &= \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{c}, \quad x \in \mathbb{R}^1, \quad a < x < b. \end{aligned}$$

Then we have the following mapping relations:

$$\begin{aligned} f(x) &= \int_0^1 S(\xi) d\xi - \frac{\pi}{c} \int_0^x \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) [S(e^z) e^z] \Big|_{z=0} dx, \\ g(x) &= \frac{\pi}{c} \int_0^x \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) [S(e^z) e^z] \Big|_{z=0} dx. \end{aligned} \quad (18)$$

And the endpoints  $a$  and  $b$  of the interval  $a < x < b$  are non-analytical points (singularities) of Fourier series, which can be uniquely determined by the detailed computation of the right-hand side of (18).

**Proof.** Combining Theorem 2 with Theorem 3 will lead us to the proof.

Apparently, such summation method of Fourier series can be extended into other trigonometric series.

**Theorem 5.** Let  $S(t) \in J$  be the sum function of power series  $\sum_{n=0}^{\infty} a_n t^n$ ,  $f(x)$  be the sum function of the corresponding  $\sum_{n=0}^{\infty} a_n \cos(nx) \cos^n x$ ,  $g(x)$  be the sum

function of the corresponding  $\sum_{n=0}^{\infty} a_n \sin(nx) \cos^n x$ , namely

$$\begin{aligned} S(t) &= \sum_{n=0}^{\infty} a_n t^n, \quad t \in \mathbb{R}^1, \quad 0 \leq t \leq r, \quad 0 < r < +\infty, \\ f(x) &= \sum_{n=0}^{\infty} a_n \cos(nx) \cos^n x, \\ g(x) &= \sum_{n=0}^{\infty} a_n \sin(nx) \cos^n x, \quad x \in \mathbb{R}^1, \quad a < x < b, \end{aligned}$$

then we have the following mapping relations:

$$\begin{aligned} f(x) &= \cos \left( x \rho \frac{\partial}{\partial \rho} \right) S(\rho) \Big|_{\rho=\cos x} = \cos \left( Y \frac{\partial}{\partial X} \right) S(X), \\ g(x) &= \sin \left( x \rho \frac{\partial}{\partial \rho} \right) S(\rho) \Big|_{\rho=\cos x} = \sin \left( Y \frac{\partial}{\partial X} \right) S(X). \end{aligned} \tag{19}$$

Where  $X = \rho \cos x = \cos^2 x$ ,  $Y = \rho \sin x = \cos x \sin x$ . And the endpoints  $a$  and  $b$  of the interval  $a < x < b$  are non-analytical points (singularities) of trigonometric series, which can be uniquely determined by the detailed computation of the right-hand side of (19).

**Proof.** According to the Definition 5 in reference [4], the following two expressions are obvious:

$$\begin{aligned} f(x) &= \cos \left( x \rho \frac{\partial}{\partial \rho} \right) S(\rho) \Big|_{\rho=\cos x}, \\ g(x) &= \sin \left( x \rho \frac{\partial}{\partial \rho} \right) S(\rho) \Big|_{\rho=\cos x}, \end{aligned}$$

where  $\cos \left( x \rho \frac{\partial}{\partial \rho} \right)$  and  $\sin \left( x \rho \frac{\partial}{\partial \rho} \right)$  are the abstract operators taking  $\rho \frac{\partial}{\partial \rho}$  as the operator element. By using the Theorem 3 in reference [4], namely

Let  $\rho \in R^n$ ,  $\theta \in R_n$ ,  $X = (\rho_1 \cos \theta_1, \dots, \rho_n \cos \theta_n)$ ,  $Y = (\rho_1 \sin \theta_1, \dots, \rho_n \sin \theta_n)$ , then for an arbitrary analytic function  $f(\rho)$ , we have

$$\begin{aligned} \cos \left( \theta \rho \frac{\partial}{\partial \rho} \right) f(\rho) &= \cos(Y \partial_X) f(X), \\ \sin \left( \theta \rho \frac{\partial}{\partial \rho} \right) f(\rho) &= \sin(Y \partial_X) f(X), \end{aligned} \tag{20}$$

where

$$\theta \rho \frac{\partial}{\partial \rho} = \left( \theta_1 \rho_1 \frac{\partial}{\partial \rho_1} + \dots + \theta_n \rho_n \frac{\partial}{\partial \rho_n} \right),$$

we can obtain (19) immediately.

**Corollary 1.** If the power series expansion of an analytic function is unique, then according to the mapping relations between analytic functions and periodic functions, the Fourier series expansion of a periodic function is unique as well.

It is easy to prove the uniqueness of power series expansions of analytic functions, thus Corollary 1 can actually derive the uniqueness of Fourier series expansions of periodic functions easier.

Theorem 2 can be directly extended into the multiple Fourier series while the form of expressions remains essentially constant, namely

**Theorem 6.** Let  $S(t) \in J(\mathfrak{D})$  be the sum function of power series  $\sum_{n=0}^{\infty} a_n t^n$ ,  $f(x) \in L^2(\Omega)$  be the sum function of the corresponding cosine series, and  $g(x) \in L^2(\Omega)$  be that of the corresponding sine series, namely

$$\begin{aligned} S(t) &= \sum_{n=0}^{\infty} a_n t^n, \quad t \in \mathfrak{D} \subset \mathbb{R}^m. \\ f(x) &= \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{c}, \\ g(x) &= \sum_{n=0}^{\infty} a_n \sin \frac{n\pi x}{c}, \quad n \in \mathbb{N}^m, \quad x \in \Omega \subset \mathbb{R}^m, \end{aligned}$$

then we have the following mapping relations:

$$\begin{aligned} f(x) &= \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}, \\ g(x) &= \sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}, \quad (z \in \mathbb{R}^m). \end{aligned} \tag{21}$$

And the non-analytical points (singularities) on the border  $\partial\Omega$  can be uniquely determined by the detailed computation of the right-hand side of (21).

Here the signs appearing in these formulas should be interpreted as the following universal abbreviation, namely

$$\begin{aligned} nx &= n_1 x_1 + n_2 x_2 + \cdots + n_m x_m, \quad a_n = a_{n_1, n_2, \dots, n_m}, \quad e^z = (e^{z_1}, e^{z_2}, \dots, e^{z_m}), \\ x \frac{\partial}{\partial z} &= x_1 \frac{\partial}{\partial z_1} + x_2 \frac{\partial}{\partial z_2} + \cdots + x_m \frac{\partial}{\partial z_m}, \quad \sum_{n=0}^{\infty} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m=0}^{\infty}. \end{aligned}$$

And the meanings of other  $t, x, t^n$  are the same with the universal signs.

**Example 1.** Trigonometric series: the sum function of  $\sum_{n=1}^{\infty} (1/n) \sin(nx) \cos^n x$  is  $g(x) = \pi/2 - x$ , and its non-analytical points are  $x = 0$  and  $x = \pi$ , namely

$$\frac{\pi}{2} - x = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx) \cos^n x, \quad 0 < x < \pi. \tag{22}$$

**Proof.** By using Theorem 5, and the algorithms and basic formulas in this paper, we have

$$\begin{aligned} S(t) &= \sum_{n=1}^{\infty} \frac{1}{n} t^n = -\ln(1-t), \quad t \in \mathbb{R}^1, \quad |t| < 1, \\ g(t) &= \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx) \cos^n x, \quad x \in \mathbb{R}^1, \quad a < x < b. \end{aligned}$$

$$\begin{aligned} g(x) &= \sin \left( x \rho \frac{\partial}{\partial \rho} \right) S(\rho) \Big|_{\rho=\cos x} = \sin \left( Y \frac{\partial}{\partial X} \right) S(X) = -\sin \left( Y \frac{\partial}{\partial X} \right) \ln(1-X) \\ &= \operatorname{arccot} \frac{1-X}{Y} = \operatorname{arccot} \frac{1-\cos^2 x}{\sin x \cos x} = \operatorname{arccot} \frac{\sin^2 x}{\sin x \cos x}. \end{aligned}$$



When  $\sin x \neq 0$ , namely  $x \neq 0$  and  $x \neq \pi$  (thus  $a = 0$ ,  $b = \pi$ ), we have

$$g(x) = \operatorname{arccot} \frac{\sin x}{\cos x} = \operatorname{arccot} \tan x = \operatorname{arccot} \cot\left(\frac{\pi}{2} - x\right) = \frac{\pi}{2} - x.$$

**Example 2.** Fourier series: the sum function of  $\sum_{n=1}^{\infty} [(-1)^{n-1}/((3n-1)(3n+1))] \cos(3n\omega t)$  is  $f(t) = (\sqrt{3}\pi/9) \cos \omega t - 1/2$ , and its non-analytical point is  $|\omega t| = \pi/3$ , namely

$$\frac{\sqrt{3}}{9} \pi \cos \omega t - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(3n-1)(3n+1)} \cos(3n\omega t), \quad -\frac{\pi}{3} < \omega t < \frac{\pi}{3}. \quad (23)$$

**Proof.** By using Theorem 2, and the algorithms and basic formulas in this paper, let

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n}}{(3n-1)(3n+1)}, \quad x \in \mathbb{R}^1, |x| < 1, \\ f(t) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(3n-1)(3n+1)} \cos(3n\omega t), \quad t \in \mathbb{R}^1, a < \omega t < b. \end{aligned}$$

As it is difficult to obtain the sum function  $S(x)$  directly, we can use the operators  $\frac{d}{dx}(x \cdot)$  and  $\frac{d}{dx}(\frac{1}{x} \cdot)$  to transform the power series into the series familiar to us, then we can obtain  $S(x)$ , and then  $f(t)$ , namely

$$\frac{d}{dx} \left( \frac{1}{x} \frac{d}{dx} (xS(x)) \right) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{3n-2} = \frac{x}{1+x^3}.$$

To describe concisely, the following results will be given directly:

$$\begin{aligned} S(x) &= \frac{1}{x} \int_0^x x \, dx \int_0^x \frac{x}{1+x^3} \, dx = \left( \frac{x}{12} - \frac{1}{12x} \right) \ln(x^2 - x + 1) \\ &\quad - \left( \frac{x}{6} - \frac{1}{6x} \right) \ln(1+x) + \left( \frac{x}{4} + \frac{1}{4x} \right) \frac{2}{\sqrt{3}} \left( \arctan \frac{2x-1}{\sqrt{3}} + \frac{\pi}{6} \right) - \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} f(t) &= \cos \left( \omega t \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \\ &= \frac{1}{6} \cos \left( \omega t \frac{\partial}{\partial z} \right) [\sinh z \ln(e^{2z} - e^z + 1)] \Big|_{z=0} \\ &\quad - \frac{1}{3} \cos \left( \omega t \frac{\partial}{\partial z} \right) [\sinh z \ln(1 + e^z)] \Big|_{z=0} \\ &\quad + \frac{1}{\sqrt{3}} \cos \left( \omega t \frac{\partial}{\partial z} \right) \left[ \cosh z \left( \arctan \frac{2e^z - 1}{\sqrt{3}} + \frac{\pi}{6} \right) \right] \Big|_{z=0} - \frac{1}{2} \\ &= -\frac{1}{6} \sin \omega t \operatorname{arccot} \frac{(2 \cos \omega t - 1) \cos \omega t}{(2 \cos \omega t - 1) \sin \omega t} + \frac{1}{3} \sin \omega t \operatorname{arccot} \frac{\cos^2(\omega t/2)}{\sin(\omega t/2) \cos(\omega t/2)} \\ &\quad + \frac{1}{2\sqrt{3}} \cos \omega t \arctan \frac{\sqrt{3}(2 \cos \omega t - 1)}{2 \cos \omega t - 1} + \frac{\pi}{6\sqrt{3}} \cos \omega t - \frac{1}{2}. \end{aligned}$$

When  $2 \cos \omega t - 1 \neq 0$ , and  $\cos(\omega t/2) \neq 0$ , namely  $|\omega t| \neq \pi/3$  and  $|\omega t| \neq \pi$ , we have

$$\begin{aligned} f(t) &= -\frac{1}{6} \sin \omega t \operatorname{arccot} \cot \omega t + \frac{1}{3} \sin \omega t \operatorname{arccot} \cot \frac{\omega t}{2} \\ &\quad + \frac{1}{2\sqrt{3}} \cos \omega t \arctan \sqrt{3} + \frac{\pi}{6\sqrt{3}} \cos \omega t - \frac{1}{2} = \frac{\sqrt{3}}{9} \pi \cos \omega t - \frac{1}{2}. \end{aligned}$$

There are four non-analytical points in the interval  $[-\pi, \pi]$ :  $\omega t = -\pi, -\pi/3, \pi/3, \pi$ , thus  $a = -\pi/3$ ,  $b = \pi/3$ . Therefore, Example 2 is proved.

## 4 The Zeta function of odd variables

**Definition 1.** Let  $S_0(t)$  be a function analytic in the neighborhood of  $t = 0$  and

$$S_0(t) = \sum_{n=1}^{\infty} a_n t^n, \quad |t| < r, \quad 0 < r < +\infty,$$

then  $S_m(t)$  is defined as

$$S_m(t) = \underbrace{\int_0^t \frac{dt}{t} \cdots \int_0^t}_{m} S_0(t) \frac{dt}{t} = \sum_{n=1}^{\infty} a_n \frac{t^n}{n^m}. \quad (24)$$

Apparently  $S_m(1)$  is the sum function of the Dirichlet series taking  $m$  as the variable.

**Lemma 1.** According to (24),  $S_m(t)$  satisfies the following recurrence relation:

$$\int_0^t S_{m-1}(t) \frac{dt}{t} = S_m(t). \quad (25)$$

**Lemma 2.** The sum function  $S_m(1)$  has the following recurrence property:

$$\cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S_m(e^z) \Big|_{z=0} = S_m(1) - \frac{\pi}{c} \int_0^x \sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S_{m-1}(e^z) \Big|_{z=0} dx. \quad (26)$$

**Proof.** Taking  $S(x) = S_{m-1}(x)/x$  in (16), it is proved by using Lemma 1. Similarly,

**Lemma 3.**  $S_m(t)$  has the following recurrence property:

$$\sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S_m(e^z) \Big|_{z=0} = \frac{\pi}{c} \int_0^x \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S_{m-1}(e^z) \Big|_{z=0} dx. \quad (27)$$

**Theorem 7.** The sum function  $S_m(1)$  has the following property:

$$\begin{aligned} &\cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S_{m-1}(e^z) \Big|_{z=0} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left( \frac{\pi x}{c} \right)^{2k} S_{m-1-2k}(1) \\ &\quad + (-1)^r \left( \frac{\pi}{c} \right)^{2r-1} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r-1} \sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S_{m-2r}(e^z) \Big|_{z=0} dx. \end{aligned} \quad (28)$$

**Proof.** We can use the mathematical induction to prove it. According to Theorem 3, it is obviously tenable when  $r = 1$  in (28). Now we inductively hypothesize that it is tenable when  $r = K$ , namely

$$\begin{aligned} & \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-1}(e^z) \Big|_{z=0} \\ &= \sum_{k=0}^{K-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} S_{m-1-2k}(1) \\ & \quad + (-1)^K \left(\frac{\pi}{c}\right)^{2K-1} \underbrace{\int_0^x dx \cdots \int_0^x}_{2K-1} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-2K}(e^z) \Big|_{z=0} dx. \end{aligned}$$

Using Lemma 3 and 2 respectively, then the above expression turns into

$$\begin{aligned} & \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-1}(e^z) \Big|_{z=0} \\ &= \sum_{k=0}^{K-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} S_{m-1-2k}(1) \\ & \quad + (-1)^K \left(\frac{\pi}{c}\right)^{2K} \underbrace{\int_0^x dx \cdots \int_0^x}_{2K} \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-2K-1}(e^z) \Big|_{z=0} dx \\ &= \sum_{k=0}^{K-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} S_{m-1-2k}(1) + (-1)^K \left(\frac{\pi}{c}\right)^{2K} S_{m-2K-1}(1) \frac{x^{2K}}{(2K)!} \\ & \quad + (-1)^{K+1} \left(\frac{\pi}{c}\right)^{2K+1} \underbrace{\int_0^x dx \cdots \int_0^x}_{2K+1} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-2K-2}(e^z) \Big|_{z=0} dx \\ &= \sum_{k=0}^K (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} S_{m-1-2k}(1) \\ & \quad + (-1)^{K+1} \left(\frac{\pi}{c}\right)^{2K+1} \underbrace{\int_0^x dx \cdots \int_0^x}_{2K+1} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-2K-2}(e^z) \Big|_{z=0} dx. \end{aligned}$$

Thus it is tenable when  $r = K + 1$ , and then Theorem 7 is proved.

Similarly we can prove the following theorems:

**Theorem 8.** The sum function  $S_m(1)$  has the following property:

$$\begin{aligned} & \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-1}(e^z) \Big|_{z=0} \\ &= \sum_{k=1}^{r-1} (-1)^{k-1} \frac{1}{(2k-1)!} \left(\frac{\pi x}{c}\right)^{2k-1} S_{m-2k}(1) \\ & \quad + (-1)^{r-1} \left(\frac{\pi}{c}\right)^{2r-1} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r-1} \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-2r}(e^z) \Big|_{z=0} dx. \end{aligned} \tag{29}$$

**Theorem 9.** The sum function  $S_m(1)$  has the following property:

$$\begin{aligned}
& \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-1}(e^z) \Big|_{z=0} \\
&= \sum_{k=1}^r (-1)^{k-1} \frac{1}{(2k-1)!} \left(\frac{\pi x}{c}\right)^{2k-1} S_{m-2k}(1) \\
&+ (-1)^r \left(\frac{\pi}{c}\right)^{2r} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-2r-1}(e^z) \Big|_{z=0} dx.
\end{aligned} \tag{30}$$

**Theorem 10.** The sum function  $S_m(1)$  has the following property:

$$\begin{aligned}
& \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-1}(e^z) \Big|_{z=0} \\
&= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} S_{m-2k-1}(1) \\
&+ (-1)^r \left(\frac{\pi}{c}\right)^{2r} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r} \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) S_{m-2r-1}(e^z) \Big|_{z=0} dx.
\end{aligned} \tag{31}$$

**Lemma 4.** In the interval  $(0, 2c)$ , we have:

$$\begin{aligned}
\sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) (-\ln(1 - e^z)) \Big|_{z=0} &= \frac{\pi}{2} - \frac{\pi x}{2c}, \quad 0 < x < 2c. \\
\sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \ln(1 + e^z) \Big|_{z=0} &= \frac{\pi x}{2c}, \quad |x| < c. \\
\cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \arctan e^z \Big|_{z=0} &= \frac{\pi}{4}, \quad |x| < c/2.
\end{aligned} \tag{32}$$

**Proof.** By using the algorithms and basic formulas in this paper, we have

$$\begin{aligned}
& \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) (-\ln(1 - e^z)) \Big|_{z=0} = -\sin\left(Y \frac{\partial}{\partial X}\right) \ln X \Big|_{z=0} = -\operatorname{arccot} \frac{X}{Y} \Big|_{z=0} \\
&= -\operatorname{arccot} \frac{1 - \cos(\pi x/c)}{-\sin(\pi x/c)} = \operatorname{arccot} \frac{\sin^2(\pi x/(2c))}{\sin(\pi x/(2c)) \cos(\pi x/(2c))}.
\end{aligned}$$

When  $\sin(\pi x/(2c)) \neq 0$  or  $x \neq 0$ ,  $x \neq 2c$ , the above expression turns into

$$\sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) (-\ln(1 - e^z)) \Big|_{z=0} = \operatorname{arccot} \tan \frac{\pi x}{2c} = \frac{\pi}{2} - \frac{\pi x}{2c}.$$

Similarly we have

$$\begin{aligned}
& \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \ln(1 + e^z) \Big|_{z=0} = \sin\left(Y \frac{\partial}{\partial X}\right) \ln X \Big|_{z=0} = \operatorname{arccot} \frac{X}{Y} \Big|_{z=0} \\
&= \operatorname{arccot} \frac{1 + \cos(\pi x/c)}{\sin(\pi x/c)} = \operatorname{arccot} \frac{\cos^2(\pi x/(2c))}{\sin(\pi x/(2c)) \cos(\pi x/(2c))}.
\end{aligned}$$

When  $\cos(\pi x/(2c)) \neq 0$  or  $|x| \neq c$ , the above expression turns into

$$\begin{aligned} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \ln(1+e^z) \Big|_{z=0} &= \operatorname{arccot} \cot \frac{\pi x}{2c} = \frac{\pi x}{2c}. \\ \cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \arctan e^z \Big|_{z=0} &= \cos\left(Y \frac{\partial}{\partial X}\right) \arctan X \Big|_{z=0} \\ &= \frac{1}{2} \arctan \frac{2X}{1-(X^2+Y^2)} \Big|_{z=0} = \frac{1}{2} \arctan \frac{2 \cos(\pi x/c)}{1-(\cos^2(\pi x/c) + \sin^2(\pi x/c))}. \end{aligned}$$

When  $\cos(\pi x/c) \neq 0$  or  $|x| \neq c/2$ , the above expression turns into

$$\cos\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) \arctan e^z \Big|_{z=0} = \frac{1}{2} \arctan \infty = \frac{\pi}{4}.$$

**Theorem 11.** In the interval  $(0, 2c)$ , we have the following Fourier series expressions:

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n^{2r}} \cos \frac{n\pi x}{c} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} \zeta(2r-2k) \\ &\quad + (-1)^r \left(\frac{\pi}{2} \frac{1}{(2r-1)!} \left(\frac{\pi x}{c}\right)^{2r-1} - \frac{1}{2} \frac{1}{(2r)!} \left(\frac{\pi x}{c}\right)^{2r}\right). \end{aligned} \tag{33}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \sin \frac{n\pi x}{c} \\ &= \sum_{k=1}^r (-1)^{k-1} \frac{1}{(2k-1)!} \left(\frac{\pi x}{c}\right)^{2k-1} \zeta(2r+2-2k) \\ &\quad + (-1)^r \left(\frac{\pi}{2} \frac{1}{(2r)!} \left(\frac{\pi x}{c}\right)^{2r} - \frac{1}{2} \frac{1}{(2r+1)!} \left(\frac{\pi x}{c}\right)^{2r+1}\right). \end{aligned} \tag{34}$$

**Proof.** According to Definition 1, if  $S_0(t) = -\ln(1-t)$ , then

$$S_{m-1}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^m}, \quad S_{m-1}(1) = \zeta(m).$$

In Theorem 7, let  $m = 2r$ , and by using Theorem 2, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n^{2r}} \cos \frac{n\pi x}{c} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} S_{2r-1-2k}(1) \\ &\quad + (-1)^r \left(\frac{\pi}{c}\right)^{2r-1} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r-1} \sin\left(\frac{\pi x}{c} \frac{\partial}{\partial z}\right) (-\ln(1-e^z)) \Big|_{z=0} dx. \end{aligned}$$

Where  $S_{2r-1-2k}(1) = \zeta(2r-2k)$ , by using Lemma 4 we have (33).

Let  $m = 2r + 1$  in Theorem 9, using Theorem 2, and considering  $S_{2r+1-2k}(1) = \zeta(2r+2-2k)$  and Lemma 4, we have (34).

**Corollary 2.** In Theorem 11, when  $r \geq 1$ , the Fourier series expression is tenable at the endpoints of the interval.

**Proof.** Observing the summation formulas of trigonometric series given by Theorem 11, if we use the integral operator  $\int_0^x dx$  to integrate both sides of (33), we can obtain (34). It means that, for any point  $x$  in the interval of convergence of summation formulas of trigonometric series, if  $x = 0$  is within the interval or at the endpoints of it, then the termwise integration of the trigonometric series by the integral operator  $\int_0^x dx$  converges uniformly to the integration of the sum function. Actually, in the theory of Fourier series, according to the integrability, if there is a certain summation formula of trigonometric series in the open interval  $a < x < b$ , then the termwise integration of the trigonometric series by the integral operator  $\int_a^x dx$ ,  $x \in [a, b]$  converges uniformly to the integration of the sum function. In other words, for a summation formula of trigonometric series in  $a < x < b$ , if we use the integral operator  $\int_a^x dx$ ,  $x \in [a, b]$  to integrate both sides of the equation, then the new summation formula of trigonometric series obtained is definitely tenable in the closed interval  $a \leq x \leq b$ . Thus Inference 3 is proved.

**Theorem 12.** The Riemann Zeta function  $\zeta(2n)$  satisfies the following recurrence formula:

$$\sum_{k=0}^{r-1} (-1)^k \frac{\pi^{2k}}{(2k+1)!} \zeta(2r-2k) = (-1)^{r-1} \frac{\pi^{2r}}{(2r+1)!} r. \quad (35)$$

**Proof** Let  $x = c$  in the second expression of Theorem 11, then it can be proved.

**Theorem 13.** The Riemann Zeta function  $\zeta(2n)$  satisfies the following recursion formula:

$$\sum_{k=0}^{r-1} (-1)^k \frac{(2\pi)^{2k}}{(2k+1)!} \zeta(2r-2k) = (-1)^{r-1} \frac{(2\pi)^{2r}}{4(2r+1)!} (2r-1). \quad (36)$$

**Proof.** Let  $x = 2c$  in the second expression of Theorem 11, then it can be proved.

**Theorem 14.** In the interval  $(0, 2c)$ , we have the following Fourier series expressions related to  $\zeta(2n+1)$ :

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \cos \frac{n\pi x}{c} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left( \frac{\pi x}{c} \right)^{2k} \zeta(2r+1-2k) \\ & \quad + (-1)^{r-1} \left( \frac{\pi}{c} \right)^{2r} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r} \ln \left( 2 \sin \frac{\pi x}{2c} \right) dx. \end{aligned} \quad (37)$$

**Proof.** Taking  $m = 2r + 1$ ,  $S_0(t) = -\ln(1-t)$  in Theorem 10, considering it is in the interval  $(0, 2c)$ , and by using the algorithms and basic formulas in this paper, we have

$$\cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) (-\ln(1-e^z)) \Big|_{z=0}$$

$$\begin{aligned}
&= -\cos\left(Y\frac{\partial}{\partial X}\right)\ln X\Big|_{z=0} = -\frac{1}{2}\ln(X^2+Y^2)\Big|_{z=0} \\
&= -\frac{1}{2}\ln\left(\left(1-\cos\frac{\pi x}{c}\right)^2 + \sin^2\frac{\pi x}{c}\right) = -\ln\left(2\sin\frac{\pi x}{2c}\right),
\end{aligned}$$

and  $S_{2r-2k}(1) = \zeta(2r+1-2k)$ , thus it is proved by using Theorem 2.

**Lemma 5.** Let  $m$  be an arbitrary natural number, then for  $\ln x$  we have the following integral formula:

$$\underbrace{\int_0^x dx \cdots \int_0^x}_{m} \ln x \, dx = \frac{x^m}{m!} (\ln x - H_m). \quad (38)$$

Where  $H_m$  are the harmonic numbers.

**Theorem 15.** Let  $r \in \mathbb{N}$  be an arbitrary natural number, and  $B_k^*$  be the Bernoulli numbers,  $B_k^*$  satisfies the following recurrence formula:

$$\sum_{k=0}^{r-1} (-1)^k \binom{2r+1}{2k+1} B_{k+1}^* = \frac{1}{2}, \quad (39)$$

then the Riemann Zeta function  $\zeta(2n+1)$  can be recursively determined by the following recurrence formula, namely

$$\begin{aligned}
\zeta(2r+1) &= \frac{2^{4r+1}}{2^{4r+1} + 2^{2r} - 1} \sum_{k=1}^{r-1} (-1)^{k-1} \frac{1}{(2k)!} \left(\frac{\pi}{2}\right)^{2k} \zeta(2r+1-2k) \\
&+ (-1)^{r-1} \frac{2^{2r+1} \pi^{2r}}{(2^{4r+1} + 2^{2r} - 1)(2r)!} \left(H_{2r} - \ln \frac{\pi}{2}\right) \\
&+ (-1)^{r-1} \frac{2^{2r} \pi^{2r}}{2^{4r+1} + 2^{2r} - 1} \sum_{k=1}^{\infty} \frac{1}{k} \frac{B_k^*}{(2r+2k)!} \left(\frac{\pi}{2}\right)^{2k},
\end{aligned} \quad (40)$$

or, equivalently,

$$\begin{aligned}
\zeta(2r+1) &= \frac{2^{4r+1}}{2^{4r+1} + 2^{2r} - 1} \sum_{k=1}^{r-1} (-1)^{k-1} \frac{1}{(2k)!} \left(\frac{\pi}{2}\right)^{2k} \zeta(2r+1-2k) \\
&+ (-1)^{r-1} \frac{2^{2r+1} \pi^{2r}}{(2^{4r+1} + 2^{2r} - 1)(2r)!} \left(H_{2r} - \ln \frac{\pi}{2}\right) \\
&+ (-1)^{r-1} \frac{2^{2r+1} \pi^{2r}}{2^{4r+1} + 2^{2r} - 1} \sum_{k=1}^{\infty} \frac{(2k)!}{4^{2k}(2r+2k)!k} \zeta(2k).
\end{aligned} \quad (41)$$

**Proof.** According to Theorem 14 and Lemma 5, in the interval  $(0, 2c)$  (if  $r \geq 1$ , then in the closed interval  $[0, 2c]$ ) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \cos \frac{n\pi x}{c} \\
&= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} \zeta(2r+1-2k) + \frac{(-1)^{r-1}}{(2r)!} \left(\frac{\pi x}{c}\right)^{2r} \ln 2
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{r-1} \left(\frac{\pi}{c}\right)^{2r} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r} \left( \ln \frac{\pi x}{2c} + \ln \frac{\sin \frac{\pi x}{2c}}{\frac{\pi x}{2c}} \right) dx \\
& = \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} \zeta(2r+1-2k) \\
& \quad + (-1)^r \frac{1}{(2r)!} \left(\frac{\pi x}{c}\right)^{2r} \left( H_{2r} - \ln \frac{\pi x}{c} \right) \\
& \quad + (-1)^{r-1} \left(\frac{\pi}{c}\right)^{2r} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r} \ln \frac{\sin \frac{\pi x}{2c}}{\frac{\pi x}{2c}} dx \\
& = \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} \zeta(2r+1-2k) \\
& \quad + (-1)^r \frac{1}{(2r)!} \left(\frac{\pi x}{c}\right)^{2r} \left( H_{2r} - \ln \frac{\pi x}{c} \right) \\
& \quad + (-1)^{r-1} \left(\frac{\pi}{c}\right)^{2r} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r} - \sum_{k=1}^{\infty} \frac{1}{2k} \frac{2^{2k} B_k^*}{(2k)!} \left(\frac{\pi x}{2c}\right)^{2k} dx \\
& = \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} \zeta(2r+1-2k) \\
& \quad + (-1)^r \frac{1}{(2r)!} \left(\frac{\pi x}{c}\right)^{2r} \left( H_{2r} - \ln \frac{\pi x}{c} \right) \\
& \quad + (-1)^r \sum_{k=1}^{\infty} \frac{1}{2k} \frac{B_k^*}{(2r+2k)!} \left(\frac{\pi x}{c}\right)^{2r+2k}.
\end{aligned}$$

Let  $x = c/2$ , considering  $\cos \frac{(2k-1)\pi}{2} = \sin k\pi = 0$ ,  $\cos \frac{2k\pi}{2} = \cos k\pi = (-1)^k$  we have

$$\begin{aligned}
-\frac{1}{2^{2r+1}} \eta(2r+1) & = \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi}{2}\right)^{2k} \zeta(2r+1-2k) \\
& \quad + (-1)^r \frac{1}{(2r)!} \left(\frac{\pi}{2}\right)^{2r} \left( H_{2r} - \ln \frac{\pi}{2} \right) \\
& \quad + (-1)^r \sum_{k=1}^{\infty} \frac{1}{2k} \frac{B_k^*}{(2r+2k)!} \left(\frac{\pi}{2}\right)^{2r+2k}.
\end{aligned}$$

Where  $\eta(2r+1)$  is the Dirichlet Eta function, defined usually by the Dirichlet series:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad (\Re(s) > 0). \quad (42)$$

Apparently

$$\eta(2r+1) = \frac{2^{2r}-1}{2^{2r}} \zeta(2r+1), \quad r \in \mathbb{N}. \quad (43)$$

Substituting (43) into the above expression, then it is proved.



**Example 3.** In Theorem 15, taking  $r = 1$ , we have

$$\zeta(3) = \frac{6\pi^2}{35} - \frac{4\pi^2}{35} \ln \frac{\pi}{2} + \frac{\pi^2}{35} \sum_{k=1}^{\infty} \frac{B_k^* \pi^{2k}}{k 4^{k-1} (2k+2)!}. \quad (44)$$

In general, of which a useful generalization is

**Theorem 16.** We have the following relation between the Riemann Zeta functions  $\zeta(2r+1)$  and  $\zeta(2r)$ , ( $r \in \mathbb{N}$ ):

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \cos \frac{n\pi x}{c} &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left( \frac{\pi x}{c} \right)^{2k} \zeta(2r+1-2k) \\ &\quad + (-1)^r \frac{1}{(2r)!} \left( \frac{\pi x}{c} \right)^{2r} \left( H_{2r} - \ln \frac{\pi x}{c} \right) \\ &\quad + (-1)^r \left( \frac{\pi x}{c} \right)^{2r} \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{2^{2k-1} (2r+2k)!} \left( \frac{x}{c} \right)^{2k}, \quad 0 \leq x \leq 2c. \end{aligned} \quad (45)$$

In 1999, by using another method in reference [6], H. M. Srivastava has obtained the following results ( $n \in \mathbb{N}$ ):

$$\begin{aligned} \zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+1} + 2^{2n} - 1} \left[ \frac{H_{2n} - \ln(\frac{1}{2}\pi)}{(2n)!} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(\frac{1}{2}\pi)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \right]. \end{aligned} \quad (46)$$

$$\begin{aligned} \zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n}(2^{2n}+1) + 2^{2n} - 1} \left[ \frac{H_{2n} - \ln(\frac{1}{3}\pi)}{(2n)!} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(\frac{1}{3}\pi)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right]. \end{aligned} \quad (47)$$

Obviously (41) and (46) are identical. By using Theorem 16 and properties of the Hurwitz Zeta function, we also obtain the (47).

The (Hurwitz's) generalized Zeta function  $\zeta(s, a)$  and the Lerch transcendent  $\Phi(z, s, a)$  are usually defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad (\Re(s) > 1; a \neq 0, -1, -2, \dots) \quad (48)$$

and

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad (|z| \leq 1; a \neq 0, -1, -2, \dots), \quad (49)$$

so that

$$\zeta(s, 1) = \zeta(s), \quad \zeta(s, 2) = \zeta(s) - 1 \quad \text{and} \quad \zeta(s, a) + \Phi(-1, s, a) = \frac{1}{2^{s-1}} \zeta(s, \frac{a}{2}), \quad (50)$$

are known to be meromorphic (that is, analytic everywhere in the complex  $s$ -plane except for a simple pole at  $s = 1$  with residue 1).

There is also the multiplication theorem (see also reference [7])

$$m^s \zeta(s) = \sum_{k=1}^m \zeta\left(s, \frac{k}{m}\right), \quad (m \in \mathbb{N}), \quad (51)$$

of which a useful generalization is

$$\sum_{k=0}^{m-1} \zeta\left(s, a + \frac{k}{m}\right) = m^s \zeta(s, ma), \quad (m \in \mathbb{N}), \quad (52)$$

so that

$$\zeta(s, 1/3) + \zeta(s, 2/3) = (3^s - 1)\zeta(s) \quad \text{and} \quad \zeta(s, 2/3) + \zeta(s, 4/3) = 3^s \zeta(s, 2) - \zeta(s), \quad (53)$$

are known to be meromorphic.

In Theorem 16, taking  $x = c/3$ , we can apply the identities (50) and (53) in order to prove the following series representations for  $\zeta(2n+1)$ :

**Theorem 17.** Let  $r \in \mathbb{N}$  be an arbitrary natural number, and  $B_k^*$  be the Bernoulli numbers, then the Riemann Zeta function  $\zeta(2n+1)$  can be recursively determined by the following recurrence formula, namely

$$\begin{aligned} \zeta(2r+1) &= \frac{2^{2r+1} 3^{2r}}{3^{2r}(2^{2r}+1) + 2^{2r}-1} \sum_{k=1}^{r-1} (-1)^{k-1} \frac{1}{(2k)!} \left(\frac{\pi}{3}\right)^{2k} \zeta(2r+1-2k) \\ &+ (-1)^{r-1} \frac{2^{2r+1} \pi^{2r}}{(3^{2r}(2^{2r}+1) + 2^{2r}-1)(2r)!} \left(H_{2r} - \ln \frac{\pi}{3}\right) \\ &+ (-1)^{r-1} \frac{4(2\pi)^{2r}}{3^{2r}(2^{2r}+1) + 2^{2r}-1} \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2r+2k)!} \frac{\zeta(2k)}{6^{2k}}, \end{aligned} \quad (54)$$

or, equivalently,

$$\begin{aligned} \zeta(2r+1) &= \frac{2^{2r+1} 3^{2r}}{3^{2r}(2^{2r}+1) + 2^{2r}-1} \sum_{k=1}^{r-1} (-1)^{k-1} \frac{1}{(2k)!} \left(\frac{\pi}{3}\right)^{2k} \zeta(2r+1-2k) \\ &+ (-1)^{r-1} \frac{2^{2r+1} \pi^{2r}}{(3^{2r}(2^{2r}+1) + 2^{2r}-1)(2r)!} \left(H_{2r} - \ln \frac{\pi}{3}\right) \\ &+ (-1)^{r-1} \frac{(2\pi)^{2r}}{3^{2r}(2^{2r}+1) + 2^{2r}-1} \sum_{k=1}^{\infty} \frac{B_k^*}{(2r+2k)!k} \left(\frac{\pi}{3}\right)^{2k}. \end{aligned} \quad (55)$$

Obviously (54) and (47) are identical.

**Proof.** Taking  $x = c/3$  in Theorem 16, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \cos \frac{n\pi}{3} &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi}{3}\right)^{2k} \zeta(2r+1-2k) \\ &+ (-1)^r \frac{1}{(2r)!} \left(\frac{\pi}{3}\right)^{2r} \left(H_{2r} - \ln \frac{\pi}{3}\right) \\ &+ (-1)^r \left(\frac{\pi}{3}\right)^{2r} \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{2^{2k-1} (2r+2k)!} \left(\frac{1}{3}\right)^{2k}. \end{aligned} \quad (56)$$

Where

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^{2r+1}} \cos \frac{n\pi}{3} &= \cos \frac{\pi}{3} + \sum_{n=1}^{\infty} \frac{1}{(3n-1)^{2r+1}} \cos \left( \frac{3n\pi}{3} - \frac{\pi}{3} \right) \\
&+ \sum_{n=1}^{\infty} \frac{1}{(3n)^{2r+1}} \cos \frac{3n\pi}{3} + \sum_{n=1}^{\infty} \frac{1}{(3n+1)^{2r+1}} \cos \left( \frac{3n\pi}{3} + \frac{\pi}{3} \right) \\
&= \frac{1}{2} - \frac{\Phi(-1, 2r+1, 2/3)}{2 \times 3^{2r+1}} - \frac{\eta(2r+1)}{3^{2r+1}} - \frac{\Phi(-1, 2r+1, 4/3)}{2 \times 3^{2r+1}} \\
&= \frac{1}{2} - \frac{2^{-2r} \zeta(2r+1, 1/3) - \zeta(2r+1, 2/3)}{2 \times 3^{2r+1}} \\
&\quad - \frac{(1-2^{-2r})\zeta(2r+1)}{3^{2r+1}} - \frac{2^{-2r} \zeta(2r+1, 2/3) - \zeta(2r+1, 4/3)}{2 \times 3^{2r+1}} \\
&= \frac{1}{2} - \frac{\zeta(2r+1, 1/3) + \zeta(2r+1, 2/3)}{2^{2r+1} 3^{2r+1}} - \frac{(1-2^{-2r})\zeta(2r+1)}{3^{2r+1}} \\
&\quad + \frac{\zeta(2r+1, 2/3) + \zeta(2r+1, 4/3)}{2 \times 3^{2r+1}} \\
&= \frac{1}{2} - \frac{(3^{2r+1}-1)\zeta(2r+1)}{2^{2r+1} 3^{2r+1}} - \frac{(1-2^{-2r})\zeta(2r+1)}{3^{2r+1}} \\
&\quad + \frac{3^{2r+1}[\zeta(2r+1)-1] - \zeta(2r+1)}{2 \times 3^{2r+1}} \\
&= \left( \frac{1}{2} + \frac{1}{2^{2r+1} 3^{2r}} - \frac{1}{2^{2r+1}} - \frac{1}{2 \times 3^{2r}} \right) \zeta(2r+1).
\end{aligned}$$

So the (54) is proved.

Connon [8], Srivastava and Tsumura [9] reported for  $\Re(s) > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cos \frac{n\pi}{3} = \frac{1}{2} (6^{1-s} - 3^{1-s} - 2^{1-s} + 1) \zeta(s), \quad (57)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cos \frac{2n\pi}{3} = \frac{1}{2} (3^{1-s} - 1) \zeta(s), \quad (58)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sin \frac{2n\pi}{3} = \sqrt{3} \left\{ \frac{3^{-s}-1}{2} \zeta(s) + 3^{-s} \zeta\left(s, \frac{1}{3}\right) \right\}, \quad (59)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cos \frac{n\pi}{2} = 2^{-s} (2^{1-s} - 1) \zeta(s), \quad (60)$$

Taking  $x = 2c/3$  in (34), we can apply the (59) in order to prove the following Corollary:

**Corollary 3.** For Hurwitz Zeta function and Riemann Zeta functions, we have following identities ( $r \in \mathbb{N}$ ):

$$\begin{aligned}
&\sqrt{3} \left[ \zeta\left(2r+1, \frac{1}{3}\right) + \frac{1-3^{2r+1}}{2} \zeta(2r+1) \right] \\
&= \sum_{k=0}^{r-1} (-1)^k \frac{(2\pi)^{2k+1}}{(2k+1)!} 3^{2r-2k} \zeta(2r-2k) + \frac{(-1)^r 2^{2r-1} \pi^{2r+1}}{(2r+1)!} (6r+1).
\end{aligned} \quad (61)$$

**Theorem 18.** For  $r \in \mathbb{N}$ , in the interval  $[-c, c]$  we have the following Fourier series expressions:

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2r}} \cos \frac{n\pi x}{c} \\ = & \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left( \frac{\pi x}{c} \right)^{2k} \eta(2r-2k) + (-1)^r \frac{1}{2} \frac{1}{(2r)!} \left( \frac{\pi x}{c} \right)^{2r}. \end{aligned} \quad (62)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2r+1}} \sin \frac{n\pi x}{c} \\ = & \sum_{k=1}^r (-1)^{k-1} \frac{1}{(2k-1)!} \left( \frac{\pi x}{c} \right)^{2k-1} \eta(2r+2-2k) \\ & + (-1)^r \frac{1}{2} \frac{1}{(2r+1)!} \left( \frac{\pi x}{c} \right)^{2r+1}. \end{aligned} \quad (63)$$

**Proof.** According to Definition 1, if  $S_0(t) = \ln(1+t)$ , then

$$S_{m-1}(t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n^m}, \quad S_{m-1}(1) = \eta(m).$$

In Theorem 7, let  $m = 2r$ , then  $S_{2r-1-2k}(1) = \eta(2r-2k)$ . By using Theorem 2 and Lemma 4, thus the (62) is proved.

In Theorem 9, let  $m = 2r+1$ , then  $S_{2r+1-2k}(1) = \eta(2r+2-2k)$ . By using Theorem 2 and Lemma 4, thus the (63) is proved.

**Theorem 19.** The Dirichlet Eta function  $\eta(2n)$  satisfies the following recurrence formula:

$$\sum_{k=0}^{r-1} (-1)^k \frac{\pi^{2k}}{(2k+1)!} \eta(2r-2k) = (-1)^{r-1} \frac{1}{2} \frac{\pi^{2r}}{(2r+1)!}. \quad (64)$$

**Proof.** Let  $x = c$  in the second expression of Theorem 18, then it can be proved.

## 5 Sum- $a_K \pi^K$ of the Dirichlet series

We have already known many Dirichlet series of integer variables have a sum similar to the one given by

$$\zeta(2n) = \frac{2^{2n-1} B_n^*}{(2n)!} \pi^{2n}, \quad (65)$$

therefore, we introduce the definition of the sum- $a_K \pi^K$ :

**Definition 2.** The sum- $a_K \pi^K$  of the Dirichlet series of integer variables are numbers like  $a_K \pi^K$ , where  $K$  is a natural number, and  $a_K$  is a rational number.

Of course we expect the Riemann Zeta function  $\zeta(2r+1)$  has the sum- $a_K \pi^K$ , but to our disappointment, the calculation formula of  $\zeta(2r+1)$  contains the infinite power series of  $\pi$ , though it converges very fast. Then we would like to ask: Whether or not  $\zeta(2r+1)$  has a sum- $a_K \pi^K$ ?

**Theorem 20.** If the Dirichlet series of integer variables denoted by  $f(m)$  has a sum- $a_K\pi^K$ , then the corresponding cosine series definitely converges to the polynomial function in a certain interval.

**Proof.** Denoting the sum of the Dirichlet series of integer variables as  $f(m)$ , namely

$$f(m) = \sum_{n=1}^{\infty} \frac{a_n}{n^m}, \quad m \in \mathbb{N}. \quad (66)$$

Especially, we have

$$f(m) = \begin{cases} \zeta(m), & a_n = 1, \\ \eta(m), & a_n = (-1)^{n-1}. \end{cases}$$

If the corresponding cosine series of  $f(m)$  has an analytic sum function in the interval  $a < 0 \leq x < b$ , we can expand this analytic function into a power series in the neighborhood of  $x = 0$ , denoted by  $P(\pi x/c)$ , namely

$$\sum_{n=1}^{\infty} \frac{a_n}{n^m} \cos \frac{n\pi x}{c} = P\left(\frac{\pi x}{c}\right), \quad a < 0 \leq x < b.$$

Differentiating both sides of the equation to obtain derivatives of all orders less than  $m$ , and let  $x = 0$ , we derive that the first several terms of coefficients of  $P(\pi x/c)$  are related to  $f(m - 2k)$ ,  $k = 0, 1, \dots, [m/2] - 1$ , namely

$$P\left(\frac{\pi x}{c}\right) = \sum_{k=0}^{[m/2]-1} (-1)^k f(m - 2k) \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} + R\left(\frac{\pi x}{c}\right).$$

If  $x = c$  is in the interval  $(a, b)$ , then differentiating at  $x = c$  we have

$$\sum_{k=1}^{[m/2]-1} (-1)^k f(m - 2k) \frac{\pi^{2k-1}}{(2k-1)!} + R'(\pi) = 0. \quad (67)$$

We would like to ask: Is  $R'(\pi)$  in (67) unique? Is there another power series  $R'(\pi)$  of  $\pi$  making (67) tenable? As we know, if the analytic function  $g(x)$  is analytic in a neighborhood of  $x = 0$ , then in a certain interval  $a \leq x \leq b$  of  $x = 0$ , we have the unique power series expansion:

$$g(x) = a_0 + a_1x + a_2x^2 + \dots, \quad a \leq x \leq b.$$

In other words, at any point  $x = x_0$  in the interval  $a \leq x \leq b$ , the power series of the function  $g(x_0)$  is unique. According to such a uniqueness of power series expansions, the power series  $R'(\pi)$  of  $\pi$  in (67) is definitely unique. Therefore, if  $R'(\pi)$  is an infinite power series of  $\pi$ , then it definitely cannot be a polynomial of  $\pi$ , vice versa.

If the Dirichlet series  $f(m)$  has a sum- $a_K\pi^K$ , according to (67),  $R'(\pi)$  definitely has the form of the sum- $a_K\pi^K$ . But unless  $R(\pi x/c)$  is a polynomial,  $R'(\pi)$  cannot be the sum- $a_K\pi^K$ , therefore,  $R(\pi x/c)$  is definitely a polynomial and cannot be an infinite power series at the same time, thus  $P(\pi x/c)$  is definitely a polynomial. Then Theorem 20 is proved.

**Theorem 21.** Let  $r \geq 1$ , and  $B_n^*$  be the Bernoulli numbers, then in the interval  $[-c, c]$ , we have the following Fourier series expansion related to  $\eta(2n + 1)$ :

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2r+1}} \cos \frac{n\pi x}{c} \\
&= \sum_{k=0}^r (-1)^k \frac{1}{(2k)!} \left( \frac{\pi x}{c} \right)^{2k} \eta(2r+1-2k) \\
& \quad + (-1)^{r+1} \sum_{n=1}^{\infty} \frac{(2^{2n}-1)B_n^*}{2n(2r+2n)!} \left( \frac{\pi x}{c} \right)^{2r+2n}.
\end{aligned} \tag{68}$$

**Proof.** According to Definition 1, if  $S_0(t) = \ln(1+t)$ , then

$$S_{m-1}(t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n^m}, \quad S_{m-1}(1) = \eta(m).$$

In Theorem 10, let  $m = 2r+1$ , using Theorem 2, and considering  $S_{2r-2k}(1) = \eta(2r+1-2k)$  we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2r+1}} \cos \frac{n\pi x}{c} \\
&= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left( \frac{\pi x}{c} \right)^{2k} \eta(2r+1-2k) \\
& \quad + (-1)^r \left( \frac{\pi}{c} \right)^{2r} \underbrace{\int_0^x dx \cdots \int_0^x}_{2r} \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) \ln(1+e^z) \Big|_{z=0} dx.
\end{aligned}$$

In the above expression, by using the algorithms and basic formulas in this paper we have

$$\begin{aligned}
& \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) \ln(1+e^z) \Big|_{z=0} = \frac{1}{2} \ln(X^2 + Y^2) \Big|_{z=0} \\
&= \frac{1}{2} \ln \left( \left( 1 + \cos \frac{\pi x}{c} \right)^2 + \sin^2 \frac{\pi x}{c} \right) = \ln \left( 2 \cos \frac{\pi x}{2c} \right) \\
&= \ln 2 - \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n^*}{2n(2n)!} \left( \frac{\pi x}{2c} \right)^{2n}, \quad |x| < c.
\end{aligned}$$

Substituting this result into the above expression, and considering  $\eta(1) = \ln 2$ , then when  $r \geq 1$ , the formula is tenable at the endpoint  $|x| = c$ , thus we have (68) immediately.

**Theorem 22.** The Riemann Zeta function  $\zeta(2n+1)$  of odd variables does not have a sum- $a_K \pi^K$ .

**Proof.** According to Theorem 20, the necessary condition for the Dirichlet Eta function  $\eta(2n+1)$  having the sum- $a_K \pi^K$  is that the corresponding cosine series takes a polynomial as its sum function in a certain interval. According to Theorem 21 and the uniqueness of power series expansions, the corresponding cosine series of  $\eta(2n+1)$  has an analytic sum function only in the interval  $[-c, c]$ , but such a sum function cannot be a polynomial, thus the Dirichlet Eta function  $\eta(2n+1)$  cannot have a sum- $a_K \pi^K$ . In addition,

$$\zeta(2n+1) = \frac{2^{2n}}{2^{2n}-1} \eta(2n+1), \quad n \in \mathbb{N}. \tag{69}$$

Therefore, the Riemann Zeta function  $\zeta(2n+1)$  of odd variables does not have a sum- $a_K \pi^K$  as well.

In precisely the same manner, we can prove the following results for the Dirichlet Beta function  $\beta(2n+1)$ :

**Corollary 5.** For  $r \in \mathbb{N}$ , in the interval  $[-c/2, c/2]$  we have the following Fourier series expressions:

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^{2r+1}} \cos \frac{(2n+1)\pi x}{c} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} \beta(2r+1-2k) + (-1)^r \frac{\pi}{4} \frac{1}{(2r)!} \left(\frac{\pi x}{c}\right)^{2r}. \end{aligned} \quad (70)$$

Where the Dirichlet Beta function is defined as

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad (\Re(s) > 0). \quad (71)$$

By using the (70) we can easily obtain

$$\sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi}{2}\right)^{2k} \beta(2r+1-2k) = (-1)^{r-1} \frac{\pi}{4} \frac{1}{(2r)!} \left(\frac{\pi}{2}\right)^{2r}, \quad (r \in \mathbb{N}). \quad (72)$$

For any positive integer  $k$ :

$$\beta(2k+1) = \frac{(-1)^k E_{2k} \pi^{2k+1}}{4^{k+1} (2k)!}, \quad (73)$$

where  $E_n$  represent the Euler numbers. By using the (72) and (73) we can easily obtain

$$\sum_{k=0}^{r-1} \binom{2r}{2k} E_{2r-2k} = -1, \quad r \in \mathbb{N}. \quad (74)$$

According to Definition 1, if  $S_0(t) = \ln \frac{1+t}{1-t}$ , then

$$S_{m-1}(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)^m}, \quad S_{m-1}(1) = \lambda(m).$$

In Theorem 7, let  $m = 2r$ , and by using Theorem 2, considering  $S_{2r-1-2k}(1) = \lambda(2r-2k)$  and

$$\sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) \left( \frac{1}{2} \ln \frac{1+e^z}{1-e^z} \right) \Big|_{z=0} = \frac{\pi}{4}, \quad 0 < x < c,$$

thus we obtain

**Corollary 6.** For  $r \in \mathbb{N}$ , in the interval  $[0, c]$  we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2r}} \cos \frac{(2n-1)\pi x}{c} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x}{c}\right)^{2k} \lambda(2r-2k) + (-1)^r \frac{\pi}{4} \frac{1}{(2r-1)!} \left(\frac{\pi x}{c}\right)^{2r-1}. \end{aligned} \quad (75)$$

Where  $\lambda(s)$  are the Dirichlet Lambda function defined by

$$\lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}, \quad (\Re(s) > 1). \quad (76)$$

Letting  $x = c/2$ , we obtain

$$\sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi}{2}\right)^{2k} \lambda(2r-2k) = \frac{(-1)^{r-1}}{2(2r-1)!} \left(\frac{\pi}{2}\right)^{2r}, \quad r \in \mathbb{N}. \quad (77)$$

Letting  $x = c/4$  in (75), we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{[\frac{n}{2}]} \frac{1}{(2n-1)^{2r}} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{\sqrt{2}}{(2k)!} \left(\frac{\pi}{4}\right)^{2k} \lambda(2r-2k) + (-1)^r \frac{\sqrt{2}}{(2r-1)!} \left(\frac{\pi}{4}\right)^{2r}. \end{aligned} \quad (78)$$

For example, letting  $r = 1, 2$  in (78), and using (77) we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{[\frac{n}{2}]} \frac{1}{(2n-1)^2} &= \frac{\pi^2}{16} \sqrt{2}, \\ \sum_{n=1}^{\infty} (-1)^{[\frac{n}{2}]} \frac{1}{(2n-1)^4} &= \frac{11\pi^4}{1536} \sqrt{2}. \end{aligned} \quad (79)$$

**Theorem 23** Let  $S(t) = \sum_{n=0}^{\infty} a_n t^n$ ,  $t \in \mathbb{R}^1$ ,  $0 \leq t \leq r$ ,  $0 < r < +\infty$ , if

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{c} &= \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}, \\ \sum_{n=0}^{\infty} a_n \sin \frac{n\pi x}{c} &= \sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \end{aligned}$$

are tenable in  $a < x < b$ ,  $x \in \mathbb{R}^1$ , then  $\forall x_0 \in [0, \frac{b-a}{2})$ ,

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x_0}{c} \cos \frac{n\pi x}{c} &= \cosh \left( x_0 \frac{\partial}{\partial x} \right) \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}, \\ \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x_0}{c} \sin \frac{n\pi x}{c} &= \cosh \left( x_0 \frac{\partial}{\partial x} \right) \sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \end{aligned} \quad (80)$$

are tenable in  $a + x_0 < x < b - x_0$ . Accordingly, for any definite value  $x \in (a, b)$ ,  $x_0$  in (80) takes values in the following interval:

$$0 < x_0 < \begin{cases} x - a, & a < x \leq (a+b)/2, \\ b - x, & b > x \geq (a+b)/2. \end{cases}$$

**Proof** Clearly

$$\sum_{n=0}^{\infty} a_n \cos \frac{n\pi x_0}{c} \cos \frac{n\pi x}{c} = \cos \left( \frac{\pi x_0}{c} \frac{\partial}{\partial z} \right) \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}$$



$$\begin{aligned}
&= \frac{1}{2} \cos \left( \frac{\pi(x-x_0)}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} + \frac{1}{2} \cos \left( \frac{\pi(x+x_0)}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \\
&= \cosh \left( x_0 \frac{\partial}{\partial x} \right) \cos \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}. \\
&\sum_{n=0}^{\infty} a_n \cos \frac{n\pi x_0}{c} \sin \frac{n\pi x}{c} = \cos \left( \frac{\pi x_0}{c} \frac{\partial}{\partial z} \right) \sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \\
&= \frac{1}{2} \sin \left( \frac{\pi(x-x_0)}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} + \frac{1}{2} \sin \left( \frac{\pi(x+x_0)}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0} \\
&= \cosh \left( x_0 \frac{\partial}{\partial x} \right) \sin \left( \frac{\pi x}{c} \frac{\partial}{\partial z} \right) S(e^z) \Big|_{z=0}.
\end{aligned}$$

These two equations are both tenable in the common interval  $a + x_0 < x < b - x_0$  of  $a < x - x_0 < b$  and  $a < x + x_0 < b$ .

For (75), using Theorem 23 we can easily obtain:

For any definite value  $x_0 \in [0, c/2)$ , we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2r}} \cos \frac{(2n-1)\pi x_0}{c} \cos \frac{(2n-1)\pi x}{c} \\
&= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \lambda(2r-2k) \cosh \left( x_0 \frac{\partial}{\partial x} \right) \left( \frac{\pi x}{c} \right)^{2k} \\
&\quad + (-1)^r \frac{1}{(2r-1)!} \frac{\pi}{4} \cosh \left( x_0 \frac{\partial}{\partial x} \right) \left( \frac{\pi x}{c} \right)^{2r-1}
\end{aligned} \tag{81}$$

are tenable in  $x_0 \leq x \leq c - x_0$ . On the contrary, for any definite value  $x \in (0, c)$ ,  $x_0$  in (81) takes value in the following interval:

$$0 \leq x_0 \leq \begin{cases} x, & 0 < x \leq c/2, \\ c-x, & c > x \geq c/2. \end{cases}$$

When  $r = 1$ , the equation above can be expressed as:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x_0}{c} \cos \frac{(2n-1)\pi x}{c} = \lambda(2) - \frac{\pi^2 x}{4c}.$$

As the equation is tenable in  $x_0 \leq x \leq c - x_0$ ,  $x$  cannot be 0 and  $c$  unless  $x_0 = 0$ . When  $x_0 = c/4 \in [0, c/2)$ , we get

$$\sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{c} = \lambda(2)\sqrt{2} - \frac{\pi^2 x}{4c}\sqrt{2}$$

is tenable in  $c/4 \leq x \leq 3c/4$ , where  $\lambda(2) = \pi^2/8$ .

If we take not  $x_0 = c/4$  but  $x = c/4 \in (0, c/2]$ , then in  $0 \leq x_0 \leq c/4$  we have

$$\sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x_0}{c} = \lambda(2)\sqrt{2} - \frac{\pi^2}{16}\sqrt{2} = \frac{\pi^2}{16}\sqrt{2}.$$

Combining these two equations into one, we have

$$\sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{c} = \begin{cases} \pi^2 \sqrt{2}/16, & 0 \leq x \leq c/4, \\ \lambda(2)\sqrt{2} - \pi^2 x \sqrt{2}/(4c), & c/4 \leq x \leq 3c/4. \end{cases}$$

In (81), if we take not  $r = 1$  but  $r = 2$ , then similarly we have

$$\begin{aligned} & \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^4} \cos \frac{(2n-1)\pi x}{c} \\ &= \frac{5\pi^4}{768} + \frac{\pi^4 x}{128c} - \frac{\pi^4 x^2}{16c^2} + \frac{\pi^4 x^3}{24c^3}, \quad \frac{c}{4} \leq x \leq \frac{3c}{4}. \end{aligned} \quad (82)$$

$$\begin{aligned} & \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^4} \cos \frac{(2n-1)\pi x_0}{c} \\ &= \frac{11\pi^4}{1536} - \frac{\pi^4}{32c^2} x_0^2, \quad 0 \leq x_0 \leq c/4. \end{aligned} \quad (83)$$

If the following Dirichlet series is denoted by  $\mathfrak{D}(2r)$ ,

$$\mathfrak{D}(2r) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^{2r}}, \quad r \in \mathbb{N}, \quad (84)$$

then by using (75) and Theorem 20, we know that  $\mathfrak{D}(2r)$  has a sum- $a_K \pi^K$ . The (79) gives  $\mathfrak{D}(4) = 11\pi^4/1536$ ,  $\mathfrak{D}(2) = \pi^2/16$ . The (82) indicates that the corresponding cosine series of  $\mathfrak{D}(4)$  converge to polynomials. By observing (83) we can find that coefficients of the polynomial functions on the right side of the equation are definitely related to  $\mathfrak{D}(4)$  and  $\mathfrak{D}(2)$ , and can be expressed as:

$$\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^4} \cos \frac{(2n-1)\pi x_0}{c} = \mathfrak{D}(4) - \frac{1}{2!} \mathfrak{D}(2) \left( \frac{\pi x_0}{c} \right)^2.$$

The result is the same as that pointed out in the proof of Theorem 20.

Generally, we can easily prove:

**Corollary 7.** For  $r \in \mathbb{N}$ , in the interval  $[0, c/4]$  we have

$$\begin{aligned} & \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^{2r}} \cos \frac{(2n-1)\pi x_0}{c} \\ &= \sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left( \frac{\pi x_0}{c} \right)^{2k} \mathfrak{D}(2r-2k), \quad 0 \leq x_0 \leq c/4. \end{aligned} \quad (85)$$

Letting  $x_0 = c/4$  in (85), we get the recurrence formula of  $\mathfrak{D}(2r)$  as

$$\sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left( \frac{\pi}{4} \right)^{2k} \mathfrak{D}(2r-2k) = \frac{1}{2} \lambda(2r). \quad (86)$$

Clearly  $\lambda(2r)$  satisfies  $\zeta(2r) = \lambda(2r) + \zeta(2r)/2^{2r}$ , then by solving this equation for  $\lambda(2r)$  and substituting it into (86), we have

$$\sum_{k=0}^{r-1} (-1)^k \frac{1}{(2k)!} \left(\frac{\pi}{4}\right)^{2k} \mathfrak{D}(2r-2k) = \frac{2^{2r}-1}{2^{2r+1}} \zeta(2r). \quad (87)$$

Using (87) or (86), we can recursively obtain the sum- $a_K \pi^K$  of  $\mathfrak{D}(2r)$ . For instance, let  $r = 3$  in (87), using the sum- $a_K \pi^K$  of  $\mathfrak{D}(2)$ ,  $\mathfrak{D}(4)$  and  $\zeta(6)$ , we can easily obtain the sum- $a_K \pi^K$  of  $\mathfrak{D}(6)$ :

$$\mathfrak{D}(6) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (-1)^{[n/2]} \frac{1}{(2n-1)^6} = \frac{361\pi^6}{491520}. \quad (88)$$

The result is the same as that obtained by (78).

In precisely the same manner, we can prove the following results:

**Corollary 8.** If the following Dirichlet series is denoted by  $\mathcal{D}(2r+1)$ , namely

$$\mathcal{D}(2r+1) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} (-1)^{[n/2]} \frac{1}{(2n+1)^{2r+1}}, \quad r \in \mathbb{N}_0, \quad (89)$$

then for  $r \in \mathbb{N}$ , in the interval  $[0, c/4]$  we have

$$\begin{aligned} & \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} (-1)^{[n/2]} \frac{1}{(2n+1)^{2r+1}} \cos \frac{(2n+1)\pi x_0}{c} \\ &= \sum_{k=0}^r (-1)^k \frac{1}{(2k)!} \left(\frac{\pi x_0}{c}\right)^{2k} \mathcal{D}(2r+1-2k), \quad 0 \leq x_0 \leq c/4. \end{aligned} \quad (90)$$

Where  $\mathcal{D}(1) = \pi/4$ , and  $\forall r \in \mathbb{N}$

$$\mathcal{D}(2r+1) = \sum_{k=0}^{r-1} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{4}\right)^{2k+1} \lambda(2r-2k) + \frac{(-1)^r}{(2r)!} \left(\frac{\pi}{4}\right)^{2r+1}. \quad (91)$$

Letting  $x_0 = c/4$  in (90), we get the recurrence formula of  $\mathcal{D}(2r+1)$  as

$$\frac{1}{2} \beta(2r+1) = \sum_{k=0}^r \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{4}\right)^{2k} \mathcal{D}(2r+1-2k), \quad r \in \mathbb{N}. \quad (92)$$

For example, letting  $r = 1, 2, 3$  in (92), and using (72) we obtain

$$\mathcal{D}(3) = \frac{3\pi^3}{128}, \quad \mathcal{D}(5) = \frac{57\pi^5}{24576}, \quad \mathcal{D}(7) = \frac{307\pi^7}{1310720}. \quad (93)$$

The results is the same as that obtained by (91).

Similarly, differentiating (90) at  $x_0 = c/4$  we get

$$\frac{1}{2} \lambda(2r) = \sum_{k=0}^{r-1} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{4}\right)^{2k+1} \mathcal{D}(2r-1-2k), \quad r \in \mathbb{N}. \quad (94)$$

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